Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Journal of Approximation Theory 131 (2004) 196–207
Journal of Approximation Theory
www.elsevier.com/locate/jat

Best constant approximants in Lorentz spaces☆

F.E. Levis*, H.H. Cuenya

Departamento de Matemática, Facultad de Cs. Exactas, Fco.-Qcas. y Naturales, Universidad Nacional de Río Cuarto, Río Cuarto, 5800, Argentina

Received 19 August 2003; received in revised form 30 July 2004; accepted in revised form 22 October 2004

Communicated by Zeev Ditzian

Abstract

In this paper, we give a characterization of best constant approximants in Lorentz spaces $L^{w,q}$, $1 \leqslant q < \infty$, and we establish a way to obtain the best constant approximants maximum and minimum. We also study monotony of the best constant approximation operator.

MSC: Primary 41A65; Secondary 41A30

Keywords: Best constant approximants; Lorentz space; Monotony

1. Introduction

Let $\mathcal{M}_0$ be the class of all real extended μ-measurable functions on $[0, 1]$, where μ is the Lebesgue measure. As usual, for $f \in \mathcal{M}_0$ we denote by

$$\mu_f(\lambda) = \mu(\{x \in [0, 1] : |f(x)| > \lambda\}), \quad (\lambda \geqslant 0),$$

its distribution function and by

$$f^*(t) = \inf\{\lambda : \mu_f(\lambda) \leqslant t\}, \quad (t \geqslant 0),$$

☆ Totally supported for Universidad Nacional de Río Cuarto. Argentina.

* Corresponding author. Fax: +54 358 4676228.
E-mail address: flevis@exa.unrc.edu.ar (F.E. Levis).

0021-9045/$ - see front matter © 2004 Elsevier Inc. All rights reserved.
doi:10.1016/j.jat.2004.10.001

its decreasing rearrangement. We recall that $\mu_f(f^*(t)) \leqslant t, t \geqslant 0$. For other properties of μ_f and f^*, the reader can see [1, p. 36–42].

Now, we give some basic notations and definitions. Let $w : (0, 1] \to (0, \infty)$ be a weight function, non-increasing and locally integrable with respect to μ. For $f \in \mathcal{M}_0$ and $1 \leqslant q < \infty$, let

$$\|f\|_{w,q} = \left(\int_0^1 w(t)(f^*(t))^q \, d\mu(t)\right)^{\frac{1}{q}}.$$

We consider the Lorentz space $L^{w,q} := \{f \in \mathcal{M}_0 : \|f\|_{w,q} < \infty\}$. For $f \in L^{w,q}$, let C_f be the set of all $c \in \mathbb{R}$ such that

$$\|f - c\|_{w,q} = \inf_{k \in \mathbb{R}} \|f - k\|_{w,q}.$$

It is well known that C_f is a non-empty and compact interval. Each element of C_f is called a best approximant of *f*. We denote $\underline{f} = \min(C_f)$ and $\overline{f} = \max(C_f)$. Consider the best approximation operator, defined by $T(f) := C_f$. In [5], Landers and Rogge, introduced the following monotony concept:

$$T \text{ is monotone iff } f \leqslant g, c \in C_f, d \in C_g \text{ then } c \vee d \in C_g \text{ and } c \wedge d \in C_f,$$

where $c \vee d = \max\{c, d\}$ and $c \wedge d = \min\{c, d\}$.

In [3] a description of the best monotone approximants, in $L^1[0, 1]$, is given. In [2] the authors gave a method to construct a best monotone approximant in $L^1[0, 1]$ and in [9], it was extended for $L^p[0, 1]$, $1 < p < \infty$. Later in [6], Marano and Quesada studied approximation in $L_\phi[0, 1]$, for a suitable function ϕ. More precisely, they gave a characterization of the best monotone approximants set and a method to construct the best monotone approximants maximum and minimum. On the other hand, Landers and Rogge in [5] studied the monotony of the best monotone approximation operator in L_ϕ.

In this paper, we shall be restricting ourself to consider simple functions almost everywhere. The motive is the difficulty in working with the Lorentz norm in approximation problems. In fact, before that an integration of the data with a certain weight be done, a non-increasing rearrangement of them is necessary. This rearrangement does not allow us, in general, to find a suitable expression for the Gateaux derivative of the norm at a given function. On the other hand, it is well known (see [7, p. 3]) that the Gateaux derivative provides a characterization of best approximants on subspaces.

In Section 2, we give a characterization of best constant approximants for a simple function and we establish a way to obtain the best constant approximants maximum and minimum.

In Section 3, we study the monotony of the best constant approximation operator *T*, in the sense of Landers and Rogge.

2. Characterization of the best constant approximants

Given a non-constant simple function h we denote $R(h) = \{h_k : 1 \leqslant k \leqslant l\}$, the range of h. We introduce the following notations

$$\delta_h = \min\{||h_i| - |h_j|| : h_i \neq h_j\}, \qquad \gamma_h = \min\{|h_i| : |h_i| > 0\},$$

$$\beta_h = \begin{cases} \frac{\min\{\delta_h, \gamma_h\}}{4} & \text{if } \delta_h > 0, \\ \frac{\gamma_h}{4} & \text{if } \delta_h = 0 \end{cases}$$

and $K_h = \min\{\beta_{h-\overline{h}}, \beta_{h-\underline{h}}\}$. Clearly $\beta_h > 0$ and $K_h > 0$. Since $(h+a) - \overline{h+a} = h - \overline{h}$ and $(h+a) - \underline{h+a} = h - \underline{h}$, $a \in \mathbb{R}$, we have

$$K_{h+a} = K_h, \quad a \in \mathbb{R}. \tag{1}$$

We denote χ_A the characteristic function of the set A.

Lemma 2.1. *Let $g \in L^{w,q}$ and let $E \subset [0,1]$ be a μ-measurable, such that $g\chi_E \geqslant 0$. Then for all $c > 0$,*

$$((g+c)\chi_E)^* = ((g\chi_E)^* + c)\chi_{[0,\mu(E))}. \tag{2}$$

Proof. From the definitions of distribution function and decreasing rearrangement, we get

$$\mu_{(g+c)\chi_E}((g\chi_E)^*(t) + c) = \mu_{g\chi_E}((g\chi_E)^*(t)) \leqslant t$$

and

$$\mu_{g\chi_E}(((g+c)\chi_E)^*(t) - c) = \mu_{(g+c)\chi_E}(((g+c)\chi_E)^*(t)) \leqslant t$$

for all $t \geqslant 0$. Therefore, the lemma is an immediate consequence of the definition of the decreasing rearrangement. □

We recall that a function $\sigma : [0,1] \to [0,1]$ is a measure preserving transformation if, whenever E is a measurable subset of $[0,1]$, the set $\sigma^{-1}(E)$ is a measurable subset of $[0,1]$ and $\mu(\sigma^{-1}(E)) = \mu(E)$.

The following theorem was proved in [1, p. 82].

Theorem (J.V. Ryff). *Let (R, μ) be a finite non-atomic measure space and let f be a non-negative μ-measurable function on R. Then there is a measure preserving transformation $\sigma : R \to [0, \mu(R)]$ such that $f = f^* \circ \sigma$ μ-a.e. on R.*

The following lemma is the key for the proof of the main theorem of this Section. We shall only work with simple functions, because the next lemma cannot be extended to every function in $L^{w,q}$. This can be seen with simple examples.

Lemma 2.2. *Let h be a simple function,* $0 < \varepsilon < \beta_h$ *and let u be a measurable function such that* $0 \leqslant u < \varepsilon$*. Then there is a measure preserving transformation* $\sigma : [0, 1] \to [0, 1]$ *such that*

$$(h + su + t)^* \circ \sigma = |h + su + t| \quad \mu - a.e. \text{ on } [0, 1] \tag{3}$$

for all $s \in \{0, 1\}$, $t \in [0, \varepsilon]$. *The set where* (3) *is satisfied, does not depend on s and t.*

Proof. Fix $t \in [0, \varepsilon]$. We denote $R(h^*) = \{h_k^* : 1 \leqslant k \leqslant l'\}$, $l' \leqslant l$, the range of h^*. We consider the following μ-measurable sets,

$$E_k^+ := \{x \in [0, 1] : h(x) = h_k^*\}, \quad 1 \leqslant k \leqslant l'$$

and

$$E_k^- := \{x \in [0, 1] : h(x) = -h_k^*\}, \quad 1 \leqslant k \leqslant l', \ \ h_k^* \neq 0.$$

By Ryff theorem, there are $\sigma_k^+ : E_k^+ \to [0, \mu(E_k^+)]$ and $\sigma_k^- : E_k^- \to [0, \mu(E_k^-)]$ measure preserving transformations such that

$$((h + u)\chi_{E_k^+})^* \circ \sigma_k^+ = |(h + u)\chi_{E_k^+}|, \quad \mu - a.e. \text{ on } E_k^+ \tag{4}$$

and

$$((h + u)\chi_{E_k^-})^* \circ \sigma_k^- = |(h + u)\chi_{E_k^-}|, \quad \mu - a.e. \text{ on } E_k^-. \tag{5}$$

Clearly, $(h + u)\chi_{E_k^+} \geqslant 0$. From Lemma 2.1, for $g = h + u$, $c = t$ and $E = E_k^+$, we obtain

$$((h + u + t)\chi_{E_k^+})^*(\sigma_k^+(x)) = ((h + u)\chi_{E_k^+})^*(\sigma_k^+(x)) + t, \quad \mu - a.e. \text{ on } E_k^+. \tag{6}$$

On the other hand, from (4) follows that

$$\begin{aligned} &((h + u)\chi_{E_k^+})^*(\sigma_k^+(x)) + t = |h(x) + u(x)| + t = |h(x) + u(x) + t|, \\ &\quad \mu - a.e. \text{ on } E_k^+. \end{aligned} \tag{7}$$

So, from (6) and (7), we have

$$((h + u + t)\chi_{E_k^+})^*(\sigma_k^+(x)) = |h(x) + u(x) + t| \quad \mu - a.e. \text{ on } E_k^+. \tag{8}$$

Since $0 \leqslant u < \varepsilon$ and $0 \leqslant t \leqslant \varepsilon$, $(-(h + u) - t)\chi_{E_k^-} = h_k^* - u - t \geqslant h_k^* - 2\varepsilon$. We also have, $\varepsilon < \beta_h \leqslant \frac{h_k^*}{4}$ for all $1 \leqslant k \leqslant l'$, $h_k^* \neq 0$. Therefore, $(-(h+u)-t)\chi_{E_k^-} \geqslant 0$. Then, from Lemma 2.1, for $g = -(h + u) - t$, $c = t$ and $E = E_k^-$, we get

$$((h + u)\chi_{E_k^-})^*(\sigma_k^-(x)) = ((h + u + t)\chi_{E_k^-})^*(\sigma_k^-(x)) + t, \quad \mu - a.e. \text{ on } E_k^-. \tag{9}$$

So, (5) and (9) imply

$$((h+u+t)\chi_{E_k^-})^*(\sigma_k^-(x)) = ((h+u)\chi_{E_k^-})^*(\sigma_k^-(x)) - t = |h(x)+u(x)| - t$$
$$= |h(x)+u(x)+t|, \quad \mu - a.e. \text{ on } E_k^-. \tag{10}$$

Furthermore,

$$((h+t)\chi_{E_k^+})^*(\sigma_k^+(x)) = h_k^* + t = |h(x)+t|, \quad \mu - a.e. \text{ on } E_k^+ \tag{11}$$

and

$$((h+t)\chi_{E_k^-})^*(\sigma_k^-(x)) = h_k^* - t = |h(x)+t|, \quad \mu - a.e. \text{ on } E_k^-. \tag{12}$$

We write $L = 2l'$ if $0 \notin R(h^*)$ and $L = 2l' - 1$ if $0 \in R(h^*)$. For $1 \leqslant k \leqslant L$, we denote

$$E_k = \begin{cases} E^+_{\frac{k+1}{2}} & \text{if k is odd,} \\ E^-_{\frac{k}{2}} & \text{if k is even} \end{cases} \quad \text{and} \quad \sigma_k = \begin{cases} \sigma^+_{\frac{k+1}{2}} & \text{if k is odd,} \\ \sigma^-_{\frac{k}{2}} & \text{if k is even.} \end{cases}$$

Thus, from (8), (10), (11) and (12), we have proved that

$$((h+su+t)\chi_{E_k})^*(\sigma_k(x)) = |h(x)+su(x)+t|, \quad \mu - a.e. \text{ on } E_k \tag{13}$$

for all $s \in \{0,1\}$, $t \in [0,\varepsilon]$ and $1 \leqslant k \leqslant L$.

Let $m_0 = 0$ and $m_k = \sum_{j=1}^k \mu(E_j)$, $1 \leqslant k \leqslant L$. Next, we prove

$$(h+su+t)^*(\sigma_k(x)+m_{k-1}) = ((h+su+t)\chi_{E_k})^*(\sigma_k(x)), \quad \mu - a.e. \text{ on } E_k \tag{14}$$

for all $s \in \{0,1\}$, $t \in [0,\varepsilon]$ and $1 \leqslant k \leqslant L$. From (13) and the assumption on u, t and s, follows that

$$\mu_{h+su+t}(((h+su+t)\chi_{E_k})^*(\sigma_k(x))) = m_{k-1}$$
$$+ \mu_{(h+su+t)\chi_{E_k}}(((h+su+t)\chi_{E_k})^*(\sigma_k(x)))$$
$$\leqslant m_{k-1} + \sigma_k(x), \quad \mu - a.e. \text{ on } E_k. \tag{15}$$

The definition of $(h+su+t)^*$ and (15) imply

$$(h+su+t)^*(\sigma_k(x)+m_{k-1}) \leqslant ((h+su+t)\chi_{E_k})^*(\sigma_k(x)), \quad \mu - a.e. \text{ on } E_k. \tag{16}$$

Now, we see the reciprocal inequality of (16). A straightforward computation shows that for $z \in [m_{k-1}, m_k)$, we have $h^*_{\frac{k+1}{2}} \leqslant (h+su+t)^*(z) < h^*_{\frac{k+1}{2}} + 2\varepsilon$ if k is odd and $h^*_{\frac{k}{2}} - 2\varepsilon < (h+su+t)^*(z) \leqslant h^*_{\frac{k}{2}}$ if k is even. Since $\mu(\sigma_k^{-1}(\{m_k\})) = 0$, for $z = m_{k-1} + \sigma_k(x) \in [m_{k-1}, m_k]$, we obtain

$$\mu_{(h+su+t)\chi_{E_k}}((h+su+t)^*(m_{k-1}+\sigma_k(x))) = -m_{k-1} + \mu_{h+su+t}$$
$$\times((h+su+t)^*(m_{k-1}+\sigma_k(x)))$$
$$\leqslant \sigma_k(x), \quad \mu - a.e. \text{ on } E_k. \tag{17}$$

Thus, (17) implies

$$((h+su+t)\chi_{E_k})^*(\sigma_k(x)) \leqslant (h+su+t)^*(\sigma_k(x)+m_{k-1}) \quad \mu-a.e. \text{ on } E_k. \quad (18)$$

Finally, by (13) and (14) the function σ defined by $\sigma(x) = \sigma_k(x)+m_{k-1}$, $x \in E_k$, $1 \leqslant k \leqslant L$, is a measure preserving transformation fulfilling (3). $\square$

It follows from a characterization theorem of best approximants (see [7, p. 3]), that

$$c \in C_f \text{ iff } \Psi_+(f-c,d) = \lim_{t\to 0^+} \frac{\|f-c+td\|_{w,q}^q - \|f\|_{w,q}^q}{t} \geqslant 0$$
$$\text{for all } d \in \mathbb{R}.$$

Now, we have

$$\Psi_+(f-c,d) = \begin{cases} d\,\Gamma_+(f-c) \text{ if } d \geqslant 0, \\ d\,\Gamma_-(f-c) \text{ if } d < 0, \end{cases}$$

where $\Gamma_+(f) = \lim_{t\to 0^+} \frac{\|f+t\|_{w,q}^q - \|f\|_{w,q}^q}{t}$ and $\Gamma_-(f) = \lim_{t\to 0^-} \frac{\|f+t\|_{w,q}^q - \|f\|_{w,q}^q}{t}$. Therefore

$$c \in C_f \quad \text{iff} \quad \Gamma_+(f-c) \geqslant 0 \quad \text{and} \quad \Gamma_-(f-c) \leqslant 0. \quad (19)$$

For $\lambda \geqslant 0$, we consider

$I_f(\lambda) := \{x \in [0,1] : f = \lambda\}$,
$a_f(\lambda) := \mu_f(\lambda) + \mu(I_f(\lambda))$ and
$b_f(\lambda) := \mu_f(\lambda) + \mu(I_{|f|}(\lambda))$.

Lemma 2.3. *Let f be a simple function and $R(f) = \{f_k : 1 \leqslant k \leqslant l\}$. Then*

$$\Gamma_+(f) = q\left(\sum_{f_k \geqslant 0} \int_{\mu_f(f_k)}^{a_f(f_k)} |f_k|^{q-1} w\,d\mu - \sum_{f_k<0} \int_{a_f(-f_k)}^{b_f(-f_k)} |f_k|^{q-1} w\,d\mu\right). \quad (20)$$

In (20), we write $|f_k|^{q-1} := 1$ if $q = 1$ and $f_k = 0$.

Proof. Let $0 < \varepsilon < \beta_f$ and $u \equiv 0$. Then by Lemma 2.2 there is a measure preserving transformation, $\sigma : [0,1] \to [0,1]$ such that

$$(f+t)^* \circ \sigma = |f+t|, \quad \mu-a.e. \text{ on } [0,1], \quad t \in [0,\varepsilon].$$

So,

$$\|f+t\|_{w,q}^q = \int_0^1 w(\sigma)|f+t|^q\,d\mu, \quad t \in [0,\varepsilon].$$

Therefore, using the Lebesgue dominated convergence theorem we obtain

$$\Gamma_+(f) = \lim_{t\to 0^+} \frac{\|f+t\|_{w,q}^q - \|f\|_{w,q}^q}{t} = \int_0^1 w(\sigma)q|f|^{q-1} sgn(f)\,d\mu. \quad (21)$$

In (21), we write $|f|^{q-1}sgn(f) := 1$ if $q = 1$ and $f = 0$.

On the other hand,

$$\begin{aligned} &\sigma : I_f(f_k) \to \left[\mu_f(f_k), a_f(f_k)\right] && \text{if } f_k \geqslant 0 \text{ and} \\ &\sigma : I_f(f_k) \to \left[a_f(-f_k), b_f(-f_k)\right] && \text{if } f_k < 0. \end{aligned} \tag{22}$$

In consequence, (21) and (22) imply (20). $\square$

Since $\Gamma_-(f) = -\Gamma_+(-f)$, from Lemma 2.3 and (19) we have the following theorem of characterization.

Theorem 2.4. *Let f be a simple function and $R(f) = \{f_k : 1 \leqslant k \leqslant l\}$. Then $c \in C_f$ iff it satisfies*

(a) $\sum_{f_k \geqslant c} \int_{\mu_{f-c}(f_k-c)}^{a_{f-c}(f_k-c)} |f_k - c|^{q-1} w\, d\mu \geqslant \sum_{f_k < c} \int_{a_{f-c}(c-f_k)}^{b_{f-c}(c-f_k)} |f_k - c|^{q-1} w\, d\mu.$

(b) $\sum_{f_k \leqslant c} \int_{\mu_{c-f}(c-f_k)}^{a_{c-f}(c-f_k)} |f_k - c|^{q-1} w\, d\mu \geqslant \sum_{f_k > c} \int_{a_{c-f}(f_k-c)}^{b_{c-f}(f_k-c)} |f_k - c|^{q-1} w\, d\mu.$

Lemma 2.5. *Let f be a simple function and $R(f) = \{f_k : 1 \leqslant k \leqslant l\}$. Then $\Gamma_+(f - x)$ and $\Gamma_-(f - x)$ are nonincreasing functions of x.*

Proof. Let $c < d$. We only show that $\Gamma_+(f - d) \leqslant \Gamma_+(f - c)$. The proof of $\Gamma_-(f - d) \leqslant \Gamma_-(f - c)$ follows from the equality $\Gamma_-(f - k) = -\Gamma_+(k - f)$. We define

$$P(u) = \sum_{f_k \geqslant u} \int_{\mu_{f-u}(f_k-u)}^{a_{f-u}(f_k-u)} q|f_k - u|^{q-1} w\, d\mu$$

and

$$Q(u) = \sum_{f_k < u} \int_{a_{f-u}(u-f_k)}^{b_{f-u}(u-f_k)} q|f_k - u|^{q-1} w\, d\mu.$$

Clearly $\Gamma_+(f - u) = P(u) - Q(u)$. It will be sufficient to prove that P is a non-increasing function and Q is non-decreasing function. First, we see that P is non-increasing. Suppose $f_k \geqslant d$, then $\mu(\{x : f(x) < 2c - f_k\}) \leqslant \mu(\{x : f(x) < 2d - f_k\})$. So,

$$\mu_{f-c}(f_k - c) \leqslant \mu_{f-d}(f_k - d).$$

Furthermore, $a_{f-u}(f_k - u) - \mu_{f-u}(f_k - u) = \mu(I_f(f_k))$ for $u \leqslant f_k$. Since w is non-increasing, $|f_k - d|^{q-1} \leqslant |f_k - c|^{q-1}$ and $\{k : f_k \geqslant d\} \subset \{k : f_k \geqslant c\}$ we get, $P(d) \leqslant P(c)$.

Now, we shall prove that Q is non-decreasing. We suppose $f_k < c$. As $\mu(\{x : f(x) \geqslant 2d - f_k\}) \leqslant \mu(\{x : f(x) > 2c - f_k\})$, we have

$$a_{f-d}(d - f_k) \leqslant \mu_{f-c}(c - f_k) \leqslant a_{f-c}(c - f_k).$$

Since $|f_k - c|^{q-1} \leqslant |f_k - d|^{q-1}$, $\{k : f_k < c\} \subset \{k : f_k < d\}$, $\mu(I_f(f_k)) = b_{f-u}(u - f_k) - a_{f-u}(u - f_k)$, for $u > f_k$, and w is non-increasing, we get $Q(c) \leqslant Q(d)$. □

In the next theorem, we establish a way to obtain the best approximants $\underline{f}$ and $\overline{f}$.

Theorem 2.6. *Let f be a simple function. Then* $\overline{f} = \max\{c : \Gamma_+(f - c) \geqslant 0\}$ *and* $\underline{f} = \min\{c : \Gamma_-(f - c) \leqslant 0\}$.

Proof. Let $s = \sup\{c : \Gamma_+(f - c) \geqslant 0\}$. By (19),

$$\Gamma_+(f - \overline{f}) \geqslant 0. \tag{23}$$

Then $\overline{f} \leqslant s$. It will be sufficient to show that $\overline{f} = s$. We suppose that $\overline{f} < s$. Then there is c, $\overline{f} < c \leqslant s$ such that

$$\Gamma_+(f - c) \geqslant 0. \tag{24}$$

From Lemma 2.5 and (19),

$$\Gamma_-(f - c) \leqslant \Gamma_-(f - \overline{f}) \leqslant 0. \tag{25}$$

So, (19), (24) and (25) imply that $c \in C_f$, which is a contradiction. Similarly, we can see that $\underline{f} = \min\{c : \Gamma_-(f - c) \leqslant 0\}$. □

3. Monotony of the best constant approximation operator

In this section, we study the monotony of the best approximation operator, T in the sense of Landers and Rogge. We begin with two lemmas.

Lemma 3.1. *Let f and g be simple functions and* $0 < \varepsilon \leqslant K_f$ *such that* $0 \leqslant g - f < \varepsilon$. *If* $c \in C_f$ *and* $d \in C_g$, *then* $c \vee d \in C_g$.

Proof. Suppose $d < c$. Let $h = f - \overline{f}$ and $u = g - f$. Clearly

$$|h + u|^q - |h|^q \leqslant |h + u + t|^q - |h + t|^q. \tag{26}$$

Furthermore, $0 < \varepsilon < \beta_h$ and $0 \leqslant u < \varepsilon$. Then, by Lemma 2.2 there is a measure preserving transformation $\sigma : [0, 1] \to [0, 1]$ such that

$$(h + su + t)^* \circ \sigma = |h + su + t|, \quad \mu - a.e. \text{ on } [0, 1] \quad s \in \{0, 1\}, \ t \in [0, \varepsilon]. \tag{27}$$

So,

$$\|h + su + t\|_{w,q}^q = \int_0^1 w(\sigma)|h + su + t|^q \, d\mu, \quad s \in \{0, 1\}, \ t \in [0, \varepsilon]. \tag{28}$$

Therefore, from (26), (27) and (28), we get

$$\|h + u\|_{w,q}^q - \|h\|_{w,q}^q \leqslant \|h + u + t\|_{w,q}^q - \|h + t\|_{w,q}^q, \quad t \in [0, \varepsilon]$$

or equivalently,

$$\frac{\left\| f - \overline{f} + t)\right\|_{w,q}^q - \left\| f - \overline{f}\right\|_{w,q}^q}{t} \leqslant \frac{\left\| g - \overline{f} + t)\right\|_{w,q}^q - \left\| g - \overline{f}\right\|_{w,q}^q}{t}, \quad t \in (0, \varepsilon].$$

In consequence $\Gamma_+(f - \overline{f}) \leqslant \Gamma_+(g - \overline{f})$, and by Theorem 2.6, $\overline{f} \leqslant \overline{g}$. Since C_g is convex and $c \leqslant \overline{f}$, we have $c \vee d \in C_g$.

If $d \geqslant c$, the lemma is obvious. □

Lemma 3.2. *Let f and g be simple functions such that* $f = \sum_{k=1}^l f_k \chi_{I_k}$ *and* $g = \chi_{I_j}$ *where* $1 \leqslant j \leqslant l$. *If* $c \in C_f$ *and* $d \in C_{f+sg}$ *for* $s \geqslant 0$, *then* $c \vee d \in C_{f+sg}$.

Proof. If $s = 0$, it is obvious. Suppose $s > 0$. We only consider the non-trivial case $c > d$. By the convexity of the set C_{f+sg}, it will be sufficient to show that the following property is verified:

For all $c \in C_f$ there is $d' \in C_{f+sg}$ such that $d' \geqslant c$.

Fix $c \in C_f$. We consider the following set

$$\mathcal{C} := \{\alpha \in [0, s] \, d' \geqslant c \ \text{ for some } \ d' \in C_{f+\alpha g}\}.$$

If $a = \sup(\mathcal{C})$, we shall show that $a \in \mathcal{C}$. Let $(\alpha_n)_{n \in \mathbb{N}} \subset \mathcal{C}$ be a sequence such that $\lim_{n\to\infty} \alpha_n = a$ and $d_{\alpha_n} \in \mathcal{C}_{f+\alpha_n g}$ with $d_{\alpha_n} \geqslant c$. Since d_{α_n} is bounded, then there is a subsequence which converges to a real number d' with $d' \geqslant c$. By simplicity we denote $(d_{\alpha_n})_{n\in\mathbb{N}}$ this subsequence. For each constant function b, we have

$$\| f + \alpha_n g - d_{\alpha_n} \| \leqslant \| f + \alpha_n g - b\| \leqslant \| f + ag - b\| + |a - \alpha_n| \|\chi_{I_j}\|.$$

Then $\| f + ag - d'\| \leqslant \| f + ag - b\|$. So, $d' \in C_{f+ag}$. Suppose $a < s$ and let h be the simple function $h = f + ag$. If h is constant, then

$$f + ag = d' \leqslant f + (a + \varepsilon)g, \quad 0 < \varepsilon < s - a.$$

Therefore $c \leqslant d' \leqslant k$, for all $k \in C_{f+(a+\varepsilon)g}$. So, $a+\varepsilon \in \mathcal{C}$. This is a contradiction. Now, suppose that h is non-constant and consider $0 < \varepsilon \leqslant \min\{K_h, s-a\}$, $u = f+(a+\varepsilon)g$ and $p \in C_u$. Clearly $0 \leqslant u-h \leqslant \varepsilon \leqslant K_h$. In consequence, from Lemma 3.1, we get $p \vee d' \in C_u$. Since $c \leqslant d' \leqslant p \vee d'$ we have $a+\varepsilon \in \mathcal{C}$. This is other contradiction. So, $a = s$. $\square$

Theorem 3.3. *The best approximation operator, T, is monotone on the set of the simple functions.*

Proof. Let f and g be simple functions, $f \leqslant g$, $c \in C_f$ and $d \in C_g$. We only consider the non-trivial case $c > d$. Without loss of the generality we denote $f = \sum_{k=1}^{l} f_k \chi_{I_k}$ and $g = \sum_{k=1}^{l} g_k \chi_{I_k}$. We define

$$G_n = \begin{cases} f & \text{if } n = 0, \\ \sum_{k=1}^{n} g_k \chi_{I_k} + \sum_{k=n+1}^{l} f_k \chi_{I_k} & \text{if } 0 < n \leqslant l. \end{cases}$$

Clearly $G_0 = f$, $G_l = g$ and

$$G_{n+1} = G_n + (g_{n+1} - f_{n+1})\chi_{I_{n+1}} \geqslant G_n \quad \text{for } 0 < n \leqslant l-1.$$

We shall prove that $\overline{G_n} \leqslant \overline{G_{n+1}}$ for all $0 < n \leqslant l-1$. In fact, if $k > \overline{G_{n+1}}$ for some $k \in C_{G_n}$, then by Lemma 3.2 we get $k \in C_{G_{n+1}}$ and this is a contradiction. Therefore $d < c \leqslant \overline{f} = \overline{G_0} \leqslant \cdots \leqslant \overline{G_l} = \overline{g}$. So, $c \in C_g$.

The proof of that $d \in C_f$, follows analogously considering $-g \leqslant -f$, $-d \in C_{-g}$ and $-c \in C_{-f}$. $\square$

Theorem 3.4. *Let f and g be functions in $L^{w,q}$ such that $f \leqslant g$, $c \in C_f$ and $d \in C_g$. Then*

(a) *If C_f is unitary, $c \vee d \in C_g$.*
(b) *If C_g is unitary, $c \wedge d \in C_f$.*

Proof. (a) The case $c \leqslant d$ is trivial. Suppose $c > d$. It is well known that all non-negative measurable function is the pointwise limit of a non-decreasing sequence of non-negative simple functions (see [8]). Let $(f_n^+)_n$, $(f_n^-)_n$, $(h_n^+)_n$, $(h_n^-)_n$ be non-negative simple function sequences such that $f_n^+ \uparrow f^+$ $f_n^- \uparrow f^-$, $h_n^+ \uparrow g^+$ and $h_n^- \uparrow g^-$. Consider $g_n^+ = h_n^+ \vee f_n^+$ and $g_n^- = h_n^- \wedge f_n^-$. Since $f^+ \leqslant g^+$ and $g^- \leqslant f^-$, then $g_n^+ \uparrow g^+$ and $g_n^- \uparrow g^-$. By Lemma 2.1 in [4], we have $(f^+ - f_n^+)^* \downarrow 0$ and $|f^+ - f_n^+| \leqslant 2|f|$. Using the Lebesgue dominated convergence theorem we obtain $\left\| f^+ - f_n^+ \right\|_{w,q} \to 0$. Analogously, $\left\| f^- - f_n^- \right\|_{w,q} \to 0$, $\left\| g^+ - g_n^+ \right\|_{w,q} \to 0$ and $\left\| g^- - g_n^- \right\|_{w,q} \to 0$. Thus,

$$\|f - f_n\|_{w,q} \to 0 \quad \text{and} \quad \|g - g_n\|_{w,q} \to 0. \tag{29}$$

Let $c_n \in C_{f_n}$ and $d_n \in C_{g_n}$. Since $f_n \leqslant g_n$, Theorem 3.3 implies that

$$c_n \wedge d_n \in C_{f_n} \quad \text{and} \quad c_n \vee d_n \in C_{g_n}.$$

The sequences $(c_n \wedge d_n)_n$ and $(c_n \vee d_n)_n$ are bounded, then there are subsequences which converge, say, to c' and d' respectively. Furthermore, $c' \leqslant d'$ and from (29), $c' \in C_f$ and $d' \in C_g$. Now, if C_f is unitary, $d < c = c' \leqslant d'$. In consequence $c \in C_g$, because C_g is a convex set.

(b) The proof is analogous. □

Remark. In the case that C_f and C_g are unitary sets, we obtain monotony in the usual sense.

The following corollary provides two important cases for which there is uniqueness of the best constant approximant.

Corollary 3.5. *Let f and g be functions in $L^{w,q}$ such that $f \leqslant g$. If (a) $1 < q < \infty$ or (b) $f, g \in C[0,1]$, then $T(f) \leqslant T(g)$*

Proof. (a) If $1 < q < \infty$, $L^{w,q}$ is a convex strictly set (see [4, Theorem 3.3]). Therefore, we have uniqueness and by Theorem 3.4, the proof is complete.

Next, assume that f is a continuous function. Suppose that C_f is not unitary and let $A := \{x \in [0,1] : \underline{f} < f(x) < \overline{f}\}$. Then $\mu(A) > 0$ and

$$\left| f - \frac{1}{2}(\underline{f} + \overline{f}) \right| < \frac{1}{2}|f - \underline{f}| + \frac{1}{2}|f - \overline{f}| \quad \text{on } A. \tag{30}$$

Thus, from (30) and Lemma 3.2 in [4], we get $(f - \frac{1}{2}(\underline{f} + \overline{f}))^* < \left(\frac{1}{2}|f - \underline{f}| + \frac{1}{2}|f - \overline{f}|\right)^*$ for some set of positive measure, B. So,

$$\left\| f - \frac{1}{2}(\underline{f} + \overline{f}) \right\|_{w,q} < \frac{1}{2} \left\| f - \underline{f} \right\|_{w,q} + \frac{1}{2} \left\| f - \overline{f} \right\|_{w,q}.$$

This is a contradiction. Thus, C_f is unitary. Now, (b) follows from Theorem 3.4. □

Remark. Finally, we observe that if $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, is a differentiable convex function, with $\phi(0) = 0$, $\phi(t) > 0$ for $t > 0$, and

$$\Psi_{w,\Phi}(f) = \int_0^\infty \phi(f^*(t))w(t)\,d\mu(t)$$

is the Orlicz–Lorentz functional, then all the results of the Sections 2 and 3 are true, if we change the Lorentz norm $\| \, \|_{w,q}$ by the functional $\Psi_{w,\Phi}$ in all place.

References

[1] C. Bennett, R. Sharpley, Interpolation of Operators, Academic Press, USA, 1988.
[2] R. Huotari, D. Legg, A. Meyerowitz, D. Townsend, The natural best L_1 approximation by nondecreasing function, J. Approx. Theory 52 (1988) 132–140.
[3] R. Huotari, A. Meyerowitz, M. Sheard, Best monotone approximations in $L_1[0, 1]$, J. Approx. Theory 47 (1986) 85–91.
[4] A. Kaminska, Some remark on Orlicz–Lorentz spaces, Z. Math. Nachr. 147 (1990) 29–38.
[5] D. Landers, L. Rogge, Best approximants in L_ϕ spaces, Z. Wahrsch. Verw. Gabiete 51 (1980) 215–237.
[6] M. Marano, J.M. Quesada, L_ϕ-approximation by nondecreasing function on the interval, Constr. Approx. 13 (1997) 177–186.
[7] A. Pinkus, On L^1-approximation, Cambridge University Press, USA, 1989.
[8] H.L. Royden, Real Analysis, The Macmillan Company, USA, 1968.
[9] J.J. Swetits, S.E. Weinstein, Construction of the best monotone approximation on $L_p[0, 1]$, J. Approx. Theory 61 (1990) 118–130.